MAT 219 (lecture 9.5)

Review of Ch 7 so far.

→ More Examples 2×2 & 3×3

- Real distinct eigenvalues.
- Complex conjugate eigenvalues.

So far in Chapter 7:

Differential System: $\underline{x}' = A\underline{x}$ with $\underline{x}(0) = \underline{k}$

1. Find eigenvalues of A
   → Characteristic equation

   Find eigenvectors of A
   → "Negative reciprocals" (if 2×2)
   → Set $x=1$ & solve (if 3×3)

2. Write general solution to differential system
   → Memorized formulas

3. Plug in initial values & get constants
   → Use fundamental matrix Ψ

Two types of problem so far:

→ Real distinct eigenvalues

→ Complex conjugate eigenvalues (only use the $a+bi$ eigenvalue)

ignore $a-bi$

- $\lambda = r$ with $\underline{v} = \xi$

  solution has $\boxed{e^{rt}\xi}$ ← Greek letter: "squiggle"

- $\lambda = a \pm bi$ with $\underline{v} = \alpha \pm \beta i$

  solution has $\boxed{e^{at}\left(\underline{\alpha}\cos bt - \underline{\beta}\sin bt\right)}$

  and $\boxed{e^{at}\left(\underline{\alpha}\sin bt + \underline{\beta}\cos bt\right)}$

EX: $\lambda = -2$ with $\underline{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

solution has $\boxed{e^{-2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$

EX: $\lambda = -2+3i$ with $\underline{v} = \begin{bmatrix} -i \\ 1+2i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} i$

solution has $\boxed{e^{-2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos 3t - \begin{bmatrix} -1 \\ 2 \end{bmatrix}\sin 3t\right)}$

and $\boxed{e^{-2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 3t + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\cos 3t\right)}$

EX: $\lambda = 4i$ with $\underline{v} = \begin{bmatrix} 1 \\ -i \\ 1+2i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} i$

solution has $\boxed{\cancelto{1}{e^{at}}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\cos 4t - \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\sin 4t\right)}$

and $\boxed{\cancelto{1}{e^{at}}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\sin 4t + \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\cos 4t\right)}$

EX: $\underline{x}' = \begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} \underline{x}$

Eigenvalues: $\det \begin{bmatrix} -4-\lambda & 2 \\ 6 & -3-\lambda \end{bmatrix} = 0$

$$\lambda^2 + 7\lambda = 0$$

$$\lambda(\lambda+7) = 0 \qquad \boxed{\lambda = 0, -7}$$

Eigenvectors:

$\lambda = 0$: $\begin{bmatrix} -4-0 & 2 \\ 6 & -3-0 \end{bmatrix} \underline{v} = \underline{0}$

$$\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} \underline{v} = \underline{0}$$

$$\underline{v} = \begin{bmatrix} 2 \\ +4 \end{bmatrix} \xrightarrow{\times 1/2} \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$\lambda = -7$: $\begin{bmatrix} -4-(-7) & 2 \\ 6 & -3-(-7) \end{bmatrix} \underline{v} = \underline{0}$

$$\begin{bmatrix} 3 & 2 \\ 6 & 3 \end{bmatrix} \underline{v} = \underline{0}$$

$$\boxed{\underline{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}}$$

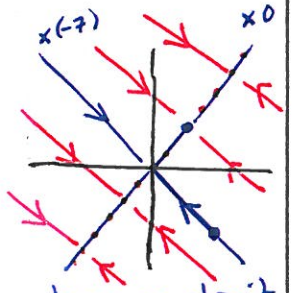
Check: $\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \stackrel{?}{=} 0 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Check: $\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \stackrel{?}{=} (-7) \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

General Solution:

$$\underline{x} = c_1 \underbrace{e^{0t}}_{1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-7t} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 2e^{-7t} \\ 2 & -3e^{-7t} \end{bmatrix}}_{\Psi} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$



phase portrait

(Ψ would be useful if we were asked to solve an initial value problem.)

EX: $\underline{x}' = \begin{bmatrix} 3 & 0 \\ 2 & -2 \end{bmatrix} \underline{x}$

Eigenvalues: $\det \begin{bmatrix} 3-\lambda & 0 \\ 2 & -2-\lambda \end{bmatrix} = 0$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda-3)(\lambda+2) = 0 \qquad \boxed{\lambda = -2, 3}$$

Eigenvectors

$\lambda = -2$: $\begin{bmatrix} 3-(-2) & 0 \\ 2 & -2-(-2) \end{bmatrix} \underline{v} = \underline{0}$

$$\begin{bmatrix} 5 & 0 \\ 2 & 0 \end{bmatrix} \underline{v} = \underline{0}$$

$$\underline{v} = \begin{bmatrix} 0 \\ -5 \end{bmatrix} \xrightarrow{\times -1/5} \boxed{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$\lambda = 3$: $\begin{bmatrix} 3-3 & 0 \\ 2 & -2-3 \end{bmatrix} \underline{v} = \underline{0}$

$$\begin{bmatrix} 0 & 0 \\ 2 & -5 \end{bmatrix} \underline{v} = \underline{0}$$

Note: cannot take negative reciprocal of top row!

$$\underline{v} = \boxed{\begin{bmatrix} +5 \\ 2 \end{bmatrix}}$$

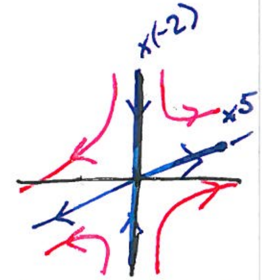
Check: $\begin{bmatrix} 3 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \stackrel{?}{=} (-2) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Check: $\begin{bmatrix} 3 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \stackrel{?}{=} 3 \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

General Solution:

$$\underline{x} = c_1 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 5e^{3t} \\ e^{-2t} & 2e^{3t} \end{bmatrix}}_{\Psi} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$



phase portrait

EX:

$$\underline{x}' = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \underline{x}$$

Eigenvalues: $\det \begin{bmatrix} -\lambda & 0 & -1 \\ 2 & -\lambda & 0 \\ -1 & 2 & 4-\lambda \end{bmatrix} = 0$

$$(-\lambda)(-\lambda)(4-\lambda) + \cancel{(0)(0)(-1)} + (-1)(2)(2)$$

$$-\left(\cancel{(-\lambda)(0)(2)} + \cancel{(0)(2)(4-\lambda)} + (-1)(-\lambda)(-1) \right) = 0$$

$$\lambda^2 \underline{\underline{(4-\lambda)}} \underline{\underline{-4+\lambda}} = 0$$

$$(\lambda^2 - 1)(4-\lambda) = 0 \qquad \boxed{\lambda = \pm 1,\ 4}$$

Eigenvectors:

$\underline{\underline{\lambda = -1}}$ $\begin{bmatrix} -(-1) & 0 & -1 \\ 2 & -(-1) & 0 \\ -1 & 2 & 4-(-1) \end{bmatrix} \underline{v} = \underline{0}$

$$\begin{bmatrix} x \quad\quad -z = 0 \\ 2x + y \quad\quad = 0 \\ -x + 2y + 5z = 0 \end{bmatrix}$$

let $\underline{\underline{x=1}}$ & solve:

$1 - z = 0$, $\underline{\underline{z = 1}}$

$2 + y = 0$, $\underline{\underline{y = -2}}$

$$\boxed{\underline{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

Check: $\begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \stackrel{?}{=} (-1) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(EX continued)

$\underline{\underline{\lambda = 1}}$ $\begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 4-1 \end{bmatrix} \underline{v} = \underline{0}$

$$\begin{bmatrix} -x \quad\quad -z = 0 \\ 2x - y \quad\quad = 0 \\ -x + 2y + 3z = 0 \end{bmatrix}$$

let $\underline{\underline{x=1}}$ & solve:

$-1 - z = 0$, $\underline{\underline{z = -1}}$

$2 - y = 0$, $\underline{\underline{y = 2}}$

$$\boxed{\underline{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}$$

Check: $\begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \stackrel{?}{=} (1) \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\underline{\underline{\lambda = 4}}$ $\begin{bmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 4-4 \end{bmatrix} \underline{v} = \underline{0}$

$$\begin{bmatrix} -4x \quad\quad -z = 0 \\ 2x - 4y \quad\quad = 0 \\ -x + 2y \quad\quad = 0 \end{bmatrix}$$

let $\underline{\underline{x=1}}$ & solve:

$-4 - z = 0$, $\underline{\underline{z = -4}}$

$2 - 4y = 0$, $\underline{\underline{y = 1/2}}$

$$\underline{v} = \begin{bmatrix} 1 \\ 1/2 \\ -4 \end{bmatrix} \xrightarrow{\times 2} \boxed{\begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}}$$

Check: $\begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix} \stackrel{?}{=} 4 \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}$

General Solution:

$$\boxed{\underline{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}}$$

$\mathbb{C}$-conjugate roots:

$\lambda = \underbrace{a} + \underbrace{b}\,i$ with $\underline{v} = \underbrace{\alpha} + \underbrace{\beta}\,i$

- a: solutions go $\begin{cases}\text{in}\\ \text{out}\\ \text{circle}\end{cases}$
- b: solutions rotate $\begin{cases}\text{fast}\\ \text{slow}\end{cases}$
- α: stretch solutions along this line
- β: solutions rotate $\begin{cases}\circlearrowright\ (+)\\ \circlearrowleft\ (-)\end{cases}$

Note: Rotation is reverse of what you may expect!

EX

$$\underline{x}' = \begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix} \underline{x}$$

Eigenvalues

$$\det \begin{bmatrix} 3-\lambda & 5 \\ -2 & 1-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 4\lambda + 13 = 0$$

$$(\lambda-2)^2 + 9 = 0 \qquad \boxed{\lambda = 2 \pm 3i}$$

Eigenvectors

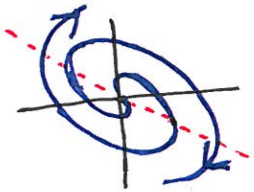
$\lambda = 2+3i$

$$\begin{bmatrix} 3-(2+3i) & 5 \\ -2 & 1-(2+3i) \end{bmatrix} \underline{v} = \underline{0}$$

$$\begin{bmatrix} 1-3i & 5 \\ -2 & -1-3i \end{bmatrix} \underline{v} = \underline{0} \qquad \boxed{\underline{v} = \begin{bmatrix} 5 \\ -(1-3i) \end{bmatrix}} = \boxed{\begin{bmatrix} 5 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} i}$$

General Solution

$$\underline{x} = c_1 e^{2t}\left(\begin{bmatrix} 5 \\ -1 \end{bmatrix} \cos 3t - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \sin 3t \right) + c_2 e^{2t}\left(\begin{bmatrix} 5 \\ -1 \end{bmatrix} \sin 3t + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \cos 3t \right)$$



phase portrait

EX:

$$\underline{x}' = \begin{bmatrix} -5 & 5 \\ -1 & -1 \end{bmatrix} \underline{x}$$

Eigenvalues

$$\det \begin{bmatrix} -5-\lambda & 5 \\ -1 & -1-\lambda \end{bmatrix} = 0$$

$$\lambda^2 + 6\lambda + 10 = 0$$

$$(\lambda+3)^2 + 1 = 0 \qquad \boxed{\lambda = -3 \pm i}$$

Eigenvectors

$\lambda = -3+i$

$$\begin{bmatrix} -5-(-3+i) & 5 \\ -1 & -1-(-3+i) \end{bmatrix} \underline{v} = \underline{0}$$

$$\begin{bmatrix} -2-i & 5 \\ -1 & 2-i \end{bmatrix} \underline{v} = \underline{0} \qquad \boxed{\underline{v} = \begin{bmatrix} 5 \\ -(-2-i) \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i}$$

General Solution:

$$\underline{x} = c_1 e^{-3t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right) + c_2 e^{-3t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t \right)$$

$$= c_1 e^{-3t} \begin{bmatrix} 5\cos t \\ 2\cos t - \sin t \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 5 \sin t \\ 2 \sin t + \cos t \end{bmatrix}$$

phase portrait

Actually... this should spiral more slowly.

EX: $\underline{x}' = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & -4 & -3 \end{bmatrix} \underline{x}$

Eigenvalues $\det \begin{bmatrix} 3-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 2 & -4 & -3-\lambda \end{bmatrix} = 0$

$(3-\lambda)(1-\lambda)(-3-\lambda) + \cancel{(0)(2)(2)} + \cancel{(0)(1)(-4)}$

$- \left(\cancel{(0)(1-\lambda)(2)} + \cancel{(0)(1)(-3-\lambda)} + (3-\lambda)(2)(-4) \right) = 0$

$\underline{\underline{(3-\lambda)}}(1-\lambda)(-3-\lambda) - \underline{\underline{(3-\lambda)}}(2)(-4) = 0$

$(3-\lambda)\left((1-\lambda)(-3-\lambda) - 2(-4)\right) = 0$

$(3-\lambda)(\lambda^2 + 2\lambda + 5) = 0$

$\boxed{\lambda = 3, \ -1 \pm 2i}$

Eigenvectors

$\underline{\underline{\lambda = 3}}$ $\begin{bmatrix} 3-3 & 0 & 0 \\ 1 & 1-3 & 2 \\ 2 & -4 & -3-3 \end{bmatrix} \underline{v} = \underline{0}$

$\begin{bmatrix} 0 = 0 \\ x - 2y + 2z = 0 \\ 2x - 4y - 6z = 0 \end{bmatrix}$

let $\underline{\underline{x=1}}$ & solve:

$(1 - 2y + 2z = 0)2$

$-(2 - 4y - 6z = 0)$

$10z = 0$

$\underline{\underline{z = 0}}$

plug in $1 - 2y + 0 = 0 \rightsquigarrow \underline{\underline{y = \tfrac{1}{2}}}$

(EX continues)

$\rightsquigarrow$ from prev. page $\lambda = 3$ has $\underline{v} = \begin{bmatrix} 1 \\ \tfrac{1}{2} \\ 0 \end{bmatrix} \xrightarrow{\times 2} \boxed{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}$

$\underline{\underline{\lambda = -1+2i}}$ $\begin{bmatrix} 3-(-1+2i) & 0 & 0 \\ 1 & 1-(-1+2i) & 2 \\ 2 & -4 & -3-(-1+2i) \end{bmatrix} \underline{v} = \underline{0}$

$\begin{bmatrix} (4-2i)x = 0 \\ x + (2-2i)y + 2z = 0 \\ 2x - 4y + (-2-2i)z = 0 \end{bmatrix}$

(**Note:** top equation says $(4-2i)x = 0$
$\Rightarrow x = 0$
so we cannot set $x = 1$!)

let $\underline{\underline{y=1}}$ & solve:

$(4-2i)x = 0 \rightsquigarrow \underline{\underline{x = 0}}$

$\cancel{x} + (2-2i)\cdot 1 + 2z = 0 \rightsquigarrow \underline{\underline{z = -1+i}}$

$\cancel{2x} - 4\cdot 1 + (-2-2i)z = 0$

$\boxed{\underline{v} = \begin{bmatrix} 0 \\ 1 \\ -1+i \end{bmatrix}} = \boxed{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} i}$

General solution: (Simplified, to save space)

$$\boxed{\underline{x} = c_1 e^{3t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ \cos t \\ -\cos t - \sin t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 0 \\ \sin t \\ -\sin t + \cos t \end{bmatrix}}$$